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Solving Share Equations in Logit Models Using the LambertW Function

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Abstract

Though individual demand and supply equations can readily be expressed in logit models, closed-form solutions for equilibrium shares and prices are intractable due to the presence of products of polynomial and exponential terms. This hinders the employment of logit models in theoretical studies, and also makes it difficult to develop reduced-form expressions for share and price as a function of exogenous variables for use in empirical studies. In this paper we propose that a mathematical function called the ‘LambertW’ be employed in solving logit models for equilibrium shares and prices. We derive closed form solutions for price and share in both the monopoly case as well as in the presence of competition. In the competitive case, the prices of the focal firm and the competitor are dependent on each other; hence the equilibrium prices are endogenous and need to be determined simultaneously. To solve this issue, we provide a simple technique that researchers can employ to derive the optimal prices for both the focal firm and the competitor simultaneously.

KEYWORDS: logit models, discrete choice models, LambertW, equilibrium prices

1. Introduction

Discrete choice models find extensive use in both the marketing and economics literature to study various aspects of consumer behavior using data on market share, price and other variables that affect demand. In spite of their widespread application in the empirical literature (Abramson, Andrews, Currim and Jones, 2000; Kamakura and Russell, 1989; Guadagni and Little, 1983; McFadden, 1978), these models remain difficult to apply in theoretical studies that require calculation of equilibrium values due to the intractable nature of the resulting expressions (for examples see Gruca, Kumar and Sudharshan, 1992; Gruca and Sudharsdhan, 1991; Karnani, 1985). This intractability arises due to the presence of products of linear and exponential expressions in the equations that need to be solved. As a result analyses are usually limited to indirect approaches that compare derivatives using the implicit function theorem, or that solve special cases (Basuroy and Nguyen, 1998; Carpenter and Lehmann, 1985; Lilien and Kotler, 1983; Lilien and Ruzdic, 1982). Hence, due to the simultaneity between price and market share, and the intractable nature of the equations, it is difficult to obtain closed form solutions of the equilibrium price in the context of the logit model.

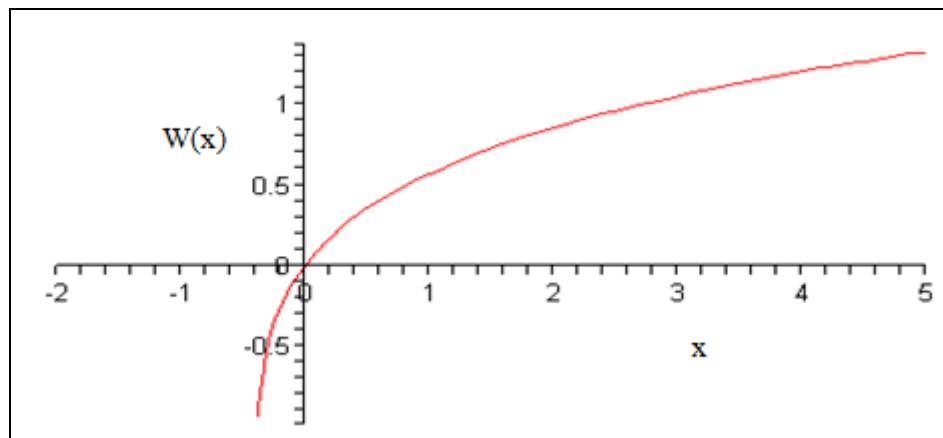
In this paper, we propose a solution to this simultaneity problem between price and market share in logit models. We employ a mathematical function (first studied by Euler, 1779), termed 'LambertW'. The LambertW function, described later in Section 2, solves several problems previously thought unsolvable. Its benefit arises from the ability to solve equations that contain products of linear and exponential expressions. The function has been employed in a variety of fields like applied mathematics, aerospace engineering, chemical engineering, physics and economics. Using the Lambert W function one can transform the price and market share equations such that they are independent of each other, thereby, yielding closed-form equilibrium solutions. We illustrate these results under two scenarios, one of a monopoly and the other of a duopoly.

The rest of the paper is organized as follows. §2 provides a brief review of the 'LambertW' function. We then describe the problems we intend to study in §3. In §4 we set up the monopoly scenario, its objective function and get the optimal closed form solutions for price and market share. A similar analysis is conducted in the duopoly case in §5, where we present the solution and the pair of equilibrium prices. We also conduct comparative statics in each case to test the impact of the parameters and certain independent variables on price. We provide our conclusions in §6.

2. LambertW Function

Lambert W is the inverse function associated with the equation— $W(x)e^{W(x)} = x$. This function belongs to the family of exponential and logarithmic functions. The forward function (shown above), resembles the exponential function and its inverse (the Lambert W) relates to the logarithmic function. Hence the shape of the LambertW function follows the shape of the exponential and logarithmic functions. The LambertW function differs from the exponential to the left of point $x = 0$. The exponential is always positive, however the LambertW dips to a minimum of -1 at $x = -1/e$. The LambertW function differs from the logarithmic function for values of $x \leq 0$, as while the logarithmic function is not defined for these values of x , the LambertW function continues to have a value till $x = -1/e$. Finally, for $-1/e < x < 0$, the inverse of the forward function described above is non-unique. The shape of the LambertW function is shown in Figure 1.

Figure 1: Plot of $W(x)$ vs. x



The importance of the Lambert W emerges from its ability to solve a series of previously unsolvable equations. Equations that involve either polynomials or exponentials are tractable and hence can be solved. Problems arise when one has to analytically solve equations containing the products of polynomials and exponentials. Given the functional form of Lambert W, many equations involving the products of both linear polynomials and exponentials can be solved using it (please refer Corless, et al., 1996 for several illustrations). This function has already seen widespread application in the fields of physics (Warburton and Wang, 2004; Valluri, Jeffrey and Corless, 2000) and applied mathematics (Corless, Jeffrey and Knuth, 1997; Jeffrey, Hare and Corless, 1996; Jeffrey, et al., 1995). In economics, this function can be used to study problems

involving continuous time lagged-dependent models (Warburton, 2004); growth models (see Croix and Licandro, 1999); models of optimal search behavior (Kauffman, Lobo and Macready, 2000); and equations that contain products of linear and exponential terms (like the logit model), to name a few. This study employs the LambertW function to solve the simultaneous equations problem of equilibrium price and market-share obtained from the logit model. Before we proceed with the analysis, we briefly outline a few important properties of the LambertW function that allow us to both, derive this solution, and manipulate it.

The derivative of LambertW is, $dW(x)/dx = 1/((1+W(x)) \cdot e^{W(x)})$, which can be simplified as $W(x)/(x \cdot (1+W(x)))$, by substituting for $e^{W(x)}$ from the forward function for LambertW. The integral of $W(x)$ is, $\int W(x)dx = x \cdot (W(x) - 1 + 1/W(x)) + C$. Finally the logarithm of $W(x)$ is given by, $\log(W(x)) = \log(x) - W(x)$. The functional form of the derivative of the Lambert W function reveals its relation to the logit model. Rearranging the terms as $x \cdot dW(x)/dx = W(x)/(1+W(x))$ and substituting $-W(x)$ for $W(x)$, the right hand side of the equation is analogous to the odds-ratio. (We thank an anonymous reviewer for highlighting this.) The next section sets up the problem we study.

3. Logit Model

We analyze two cases of the logit model. First, we illustrate the logit model for the case of a monopoly, where the consumer has to choose between the monopoly product and an outside good. Second, we study two firms that compete with each other for share in a given market.

We consider a monopolist i that sells a product i in a given market. The consumers' utility from purchasing product i is given by

$$u_i = \beta_{0i} - \beta_i P_i + \varepsilon_i = U_i + \varepsilon_i \quad (1)$$

where u_i is the utility obtained by the consumer from product i , β_{0i} is the brand or product specific parameter, β_i is the price response parameter for product i , P_i is the price of i , U_i is the deterministic component of utility that is assumed to be constant across consumers. ε_i is the random error term assumed to follow iid type 1 extreme value distribution. Given this utility formulation, the logit model computes the share of firm i in the market relative to the outside good. Normalizing the utility of the outside good to zero, i.e. $U_0 = 0$, the choice share for a monopolist i with respect to the outside good is

$$S_i = \frac{e^{U_i}}{1 + e^{U_i}} \quad (2)$$

where S_i defines the probability of choosing product i .

Similarly, we can derive the logit formulation in the case of competition. Assume two firms i and k that each sell a similar product and compete for share in a given market. Each firm sells a single product, with firm i selling product i and firm k selling product k . For the purposes of this paper, we restrict our analysis to the focal firm i . Hence, equation (1) defines the utility function for a consumer given a purchase from firm i . The share of i relative to the competing brand k and an outside good (utility of the outside good is normalized to 0; i.e. $U_0 = 0$) is,

$$S_{ic} = \frac{e^{U_i}}{1 + e^{U_i} + e^{U_k}} \quad (3)$$

where S_{ic} represents the market share for firm i in the competitive setting and U_k the utility derived from choosing the competitor's product k . This utility is represented as $u_k = \beta_{0k} - \beta_k P_k + \varepsilon_k = U_k + \varepsilon_k$, where u_k is the utility obtained by the consumer from product k , β_{0k} is the brand or product specific parameter, β_k is the price response parameter for product k , P_k is the price of k , U_k is the deterministic component of utility that is assumed to be constant across consumers and ε_k is the random error term. We also assume that the error term in the utility for k follows an iid type 1 extreme value distribution. The next section derives the optimal price equation in the monopoly setting.

4. Monopoly Setting

For the monopolist, we define firm i 's profit as follows:

$$\pi_i = (P_i - C_i)S_i \quad (4)$$

where π_i is the profit firm i earns from selling the product at price P_i and C_i , the marginal cost it incurs to provide the product. We assume a market size normalized to one. If a pure strategy interior equilibrium exists, then the price vector should satisfy the first order conditions. Hence differentiating (4) with respect to P_i , and setting the result as equivalent to zero, we get the optimal price that the firm must charge to maximize profit.

$$P_i^* = \frac{I + \beta_i C_i (1 - S_i)}{\beta_i (1 - S_i)} \quad (5)$$

where P_i^* is the optimal price charged by the firm. This optimal price is a function of share, which, in turn (as seen in equation (2)), is a function of price. The intractable nature of the market share and optimal price equations is apparent due to the presence of products of linear and exponential terms. However, applying the LambertW transformation on the two equations, we can derive the closed form solutions to price and market share by solving equations (5) and (2) simultaneously. The resulting expressions are presented in Proposition 1.

Proposition 1: Given the logit formulation for market share, the optimal price that a monopolist can charge is

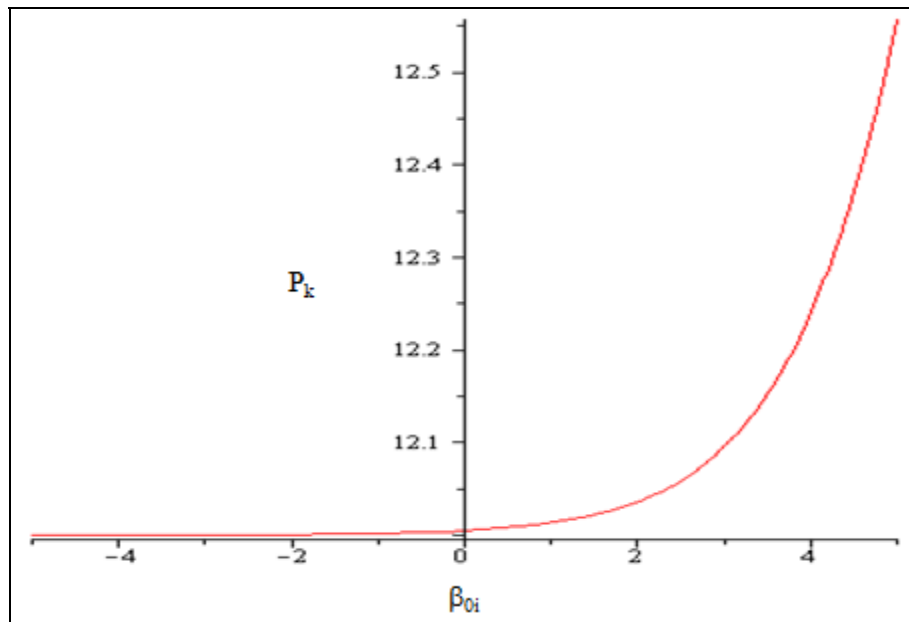
$$P_i^* = \frac{I + W(e^{\beta_{0i} - I - C_i \beta_i})}{\beta_i} + C_i \quad (6)$$

and the corresponding market share is given by

$$S_i^* = \frac{W(e^{\beta_{0i} - I - C_i \beta_i})}{I + W(e^{\beta_{0i} - I - C_i \beta_i})} \quad (7)$$

Proof: Appendix A furnishes the proof.

Equations (6) and (7) provide the general closed form solutions for price and market share in terms of LambertW that apply to any model with the structure given in equation (2) and objective function given in equation (4). Additionally, equation (6) also provides the margin the firm can obtain, given by $(I + W(e^{\beta_{0i} - I - C_i \beta_i})) / \beta_i$.

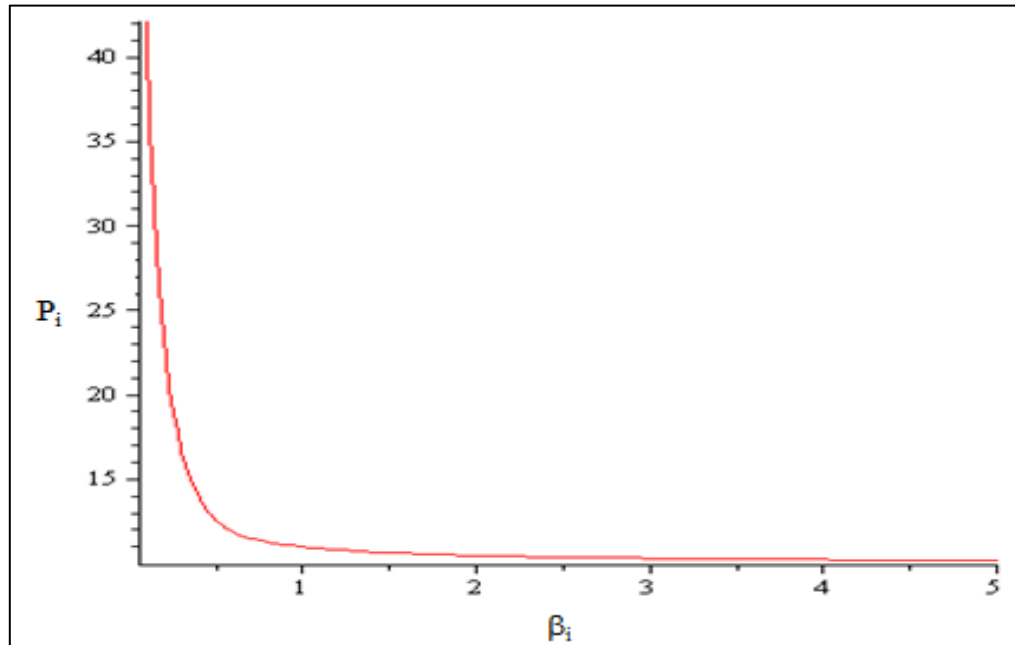
Figure 2: Plot of P_i vs. β_{0i} 

As shown above, the optimal price that the firm charges no longer depends on its own market share. These reduced-form expressions for price and share, allow the direct determination of changes in optimal price resulting from changes made to the cost or other independent variables. Additionally, employing the logit model solution in equations (6) and (7) has two major advantages. First, it facilitates numerical solution for model parameters: routines for solving LambertW function are provided in MATLAB and other commonly-used software packages. Second, it simplifies the calculation of comparative statics. For example, we can draw direct inferences on how the changes in product specific parameter β_{0i} impact the equilibrium price. To do so, we differentiate equation (6) with respect to β_{0i} and obtain $\partial P_i^* / \partial \beta_{0i} = W(e^{\beta_{0i}-1-C_i\beta_i}) / (1+W(e^{\beta_{0i}-1-C_i\beta_i}))\beta_i$, (see Appendix A for derivation) showing that price increases as β_{0i} increases, which is verified when we plot the variation of price with respect to the product specific parameter (assuming $C_i = 10$ and $\beta_i = 0.5$) in Figure (2).

Alternately, the effects of changes in the price response parameter on the equilibrium price can also be derived. Again, differentiating (6) with respect to β_i , we obtain $\frac{\partial P_i^*}{\partial \beta_i} = -\frac{C_i W(e^{\beta_{0i}-1-C_i\beta_i})}{(1+W(e^{\beta_{0i}-1-C_i\beta_i}))\beta_i} - \frac{1+W(e^{\beta_{0i}-1-C_i\beta_i})}{\beta_i^2}$, (see Appendix A for

derivation) indicating that price decreases as β_i increases, as one would expect; and on plotting, we see in Figure (3) the variation in price with respect to the price response parameter (assuming $C_i = 10$ and $\beta_{0i} = 5$). Thus expressing the logit model solution in LambertW form simplifies the process of comparative statics.

Figure 3: Plot of P_i vs. β_i



In general, employing this form allows the researcher to draw conclusions about changes in price without influencing the market share of the firm. The analysis carried out above adapts itself to testing the effects of any independent variable that might affect the utility of the consumer and in turn the equilibrium price (e.g. marketing expenditure, changes in product design etc.). The next section extends the monopoly setting to one of a duopoly.

5. Duopoly Setting

In a duopoly setting, the focal firm i competes for market share with another firm k that also sells a product similar to the focal firm. Additionally, we also allow consumers the option of not choosing any of the two products, in which case we normalize the utility to zero. Given this scenario, equation (3) (i.e.,

$S_{ic} = \frac{e^{U_i}}{1 + e^{U_i} + e^{U_k}}$) defines the market share that firm i can achieve. We then characterize the profit for firm i as

$$\pi_{ic} = (P_{ic} - C_{ic})S_{ic} \quad (8)$$

where π_{ic} corresponds to the profit firm i earns from selling the product at price P_{ic} in a competitive setting, and C_{ic} , the marginal cost it incurs to provide the product. We again assume a market size normalized to one. Similar to the monopoly case, if a pure strategy interior equilibrium exists, then the price vector should satisfy the first order conditions. Thus, differentiating the profit equation with respect to P_{ic} , and setting the result as equivalent to zero, we derive the optimal price that firm i charges to maximize profit, as shown in equation (9) which follows,

$$P_{ic}^* = \frac{1 + \beta_i C_{ic} (1 - S_{ic})}{\beta_i (1 - S_{ic})} \quad (9)$$

Both equation (9) and (3) are intractable, hence they do not readily yield closed form solutions. Additionally, similar to the monopoly case, we cannot readily draw inferences about the effects of the price response parameter on the price that firm i charges, as doing so changes the market share of the firm and in turn affects the price. Hence, employing a similar procedure as used in the monopoly case, we apply the LambertW transformation and solve the two equations (9) and (3) simultaneously to derive the optimal price and market share equations in the competitive setting. This allows direct estimation of the impact of various independent variables on the price P_{ic} without any change in the market share S_{ic} . The resulting expressions are given in Proposition 2.

Proposition 2: Given the logit formulation for market share in a duopoly setting, the optimal price that a firm can charge is

$$P_{ic}^* = \frac{1 + W\left(\frac{e^{\beta_{0i} - 1 - C_{ic}\beta_i}}{1 + e^{\beta_{0k} - P_k\beta_k}}\right)}{\beta_i} + C_{ic} \quad (10)$$

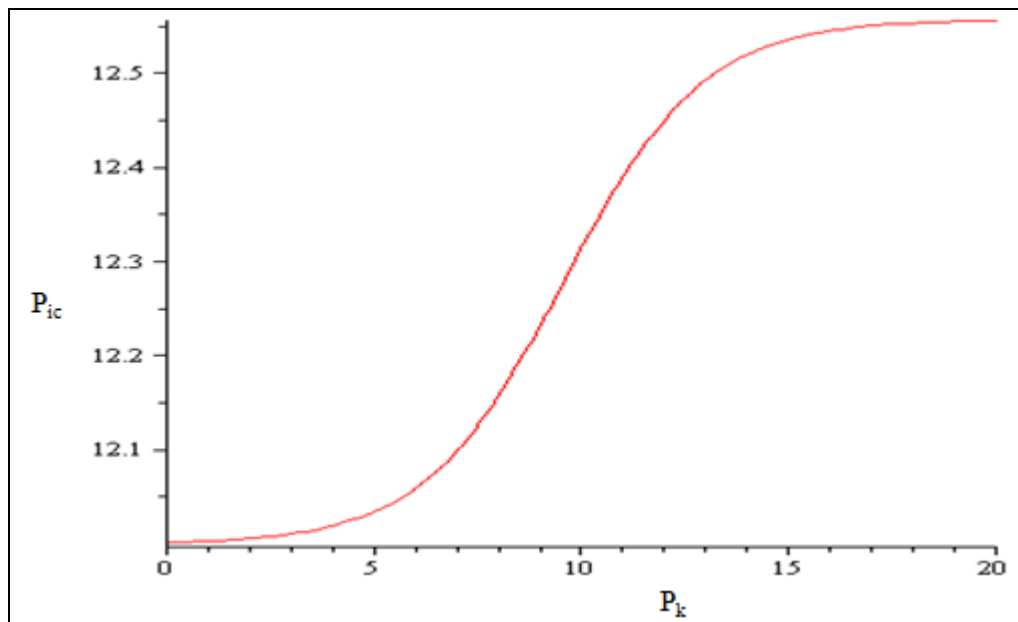
and the resultant market share is

$$S_{ic} = \frac{W\left(\frac{e^{\beta_{0i}-I-C_{ic}\beta_i}}{I + e^{\beta_{0k}-P_k\beta_k}}\right)}{I + W\left(\frac{e^{\beta_{0i}-I-C_{ic}\beta_i}}{I + e^{\beta_{0k}-P_k\beta_k}}\right)} \quad (11)$$

Proof: Appendix A furnishes the proof

Equations (10) and (11) are the closed form solutions of price and market share of firm i that are independent of each other. Hence, by solving the simultaneity problem between price and market share of firm i , inferences about the changes in price of firm i given changes in independent variables that affect it are now possible (e.g., the competitor's prices or the focal firm's cost). Additionally, given the solution to price in (10) we can track the variation of the price of the focal firm when the competitor's price changes. Figure (4) illustrates this for values of $\beta_{0i} = 5$, $\beta_i = 0.5$, $C_{ic} = 10$, $\beta_{0k} = 6$, $\beta_k = 0.6$ and $C_k = 12$.

Figure 4: Plot of P_{ic} vs. P_k



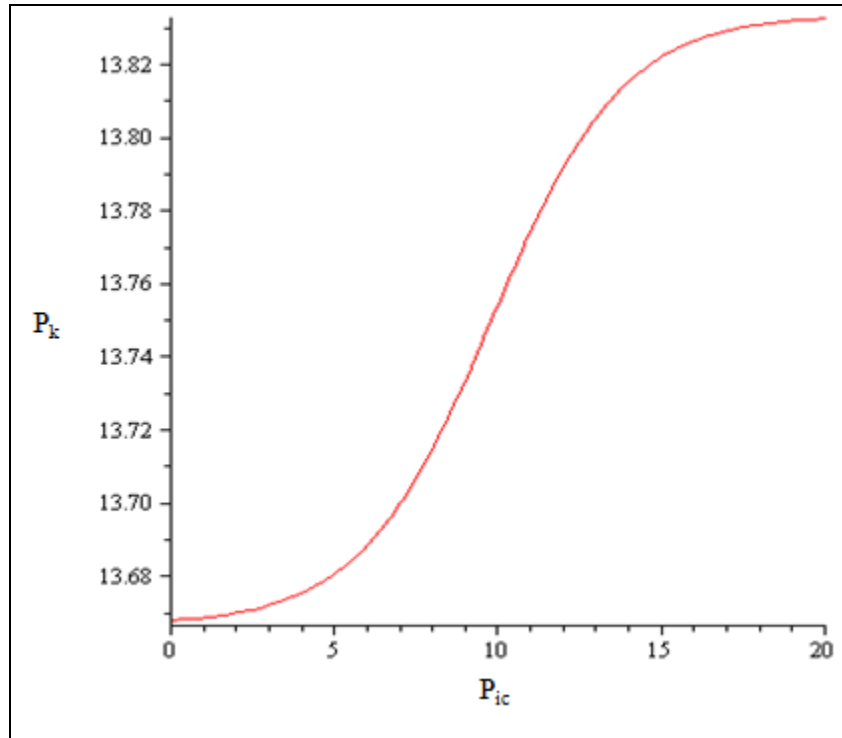
While the above equations provide separate solutions for prices and shares of firm i , the price charged by i is still dependent on firm k 's price (see equation 10). The problem of determining equilibrium market prices remains. Since P_{ic} and

P_k are symmetric, we derive the price that firm k charges in the same way as the price of firm i , and we express this below,

$$P_k^* = \frac{I + W\left(\frac{e^{\beta_{0k} - I - C_k \beta_k}}{I + e^{\beta_{0i} - P_{ic} \beta_i}}\right)}{\beta_k} + C_k \quad (12)$$

Figure (5) illustrates the variation of P_k^* given change in the price P_{ic} .

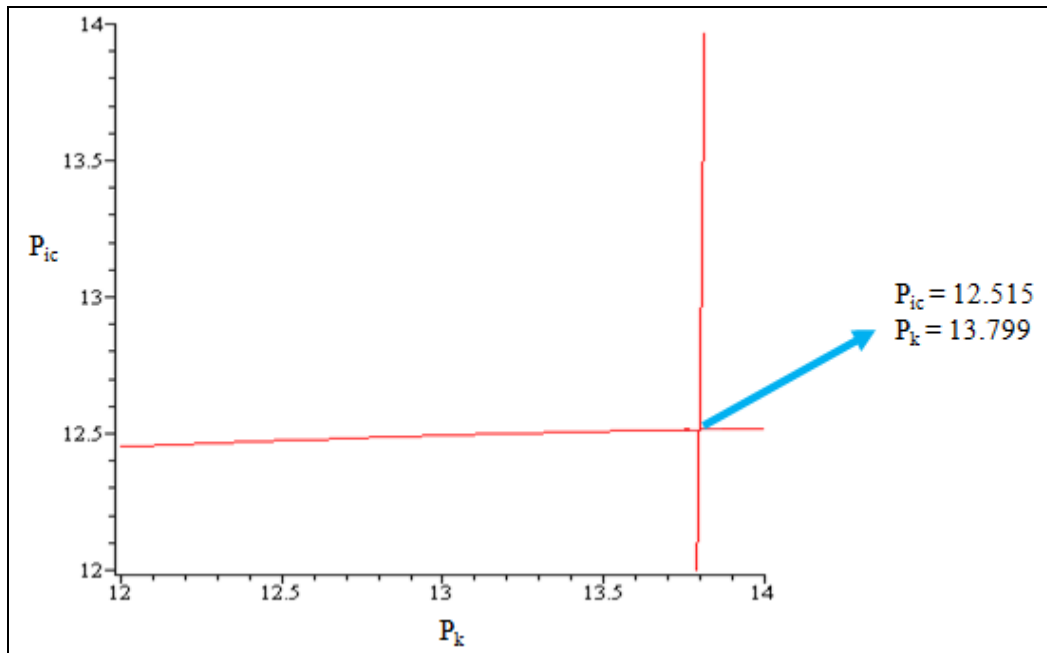
Figure 5: Plot of P_k vs. P_{ic}



If the price pair $\{P_{ic}^*, P_k^*\}$ qualifies as a Nash equilibrium, the firms' prices must satisfy equations (10) and (12). Assuming values for the parameters and the costs of each firm, the point of intersection of the curves P_{ic} given P_k and of P_k given P_{ic} gives the equilibrium prices. If we assume that $\beta_{0i} = 5$, $\beta_i = 0.5$, $C_{ic} = 10$, $\beta_{0k} = 6$, $\beta_k = 0.6$ and $C_k = 12$, then Nash equilibrium prices are $P_{ic} = 12.515$ and $P_k = 13.799$. A pseudo-code describing the solution procedure is provided in Appendix B. Figure 6 provides a graphical illustration of the plots of P_{ic} given P_k and of P_k given P_{ic} . Since the LambertW function is implemented in commonly-

used software packages, numerical solution for the equilibrium prices is straightforward. Hence, using the procedure developed in the paper, researchers can numerically evaluate the equilibrium prices given values of the parameters and independent variables.

Figure 6: The Nash Equilibrium Prices



6. Conclusions

In this paper, we analytically solved the problem of simultaneity in discrete choice models using the LambertW function. We apply the LambertW function to derive the closed form solutions to price and share both in the case of a monopoly and a duopoly. We then illustrate the changes in price with respect to several independent variables. This simplifies the process of doing comparative statics, allowing researchers the opportunity to study the effects of several independent variables on the variable of interest. Since software for evaluating the LambertW function is readily available, our methodology also facilitates the solution for equilibrium values in competitive settings. For example, equations (10) and (12) can be solved for pairs of prices and shares that satisfy the Nash equilibrium. In sum, if the relevant equations can be manipulated into the LambertW form, the calculation of numerical solutions and derivatives is likely to be simplified.

The Lambert W approach to the logit model can be extended to study markets with more than two competitors, and to determining the effect of marketing variables such as ad expenditures and promotions (examples of such models include Basuroy and Nguyen, 1998; Besanko, Gupta and Jain, 1998; Carpenter and Lehmann, 1985). Aside from the application to the logit model, the general approach outlined in this paper is potentially applicable to many scenarios, provided the equations can be manipulated into the LambertW form. Other applications can involve the study of continuous-time lagged models, where the resulting expressions would no longer remain as ordinary differential equations (example Warburton, 2004). Future research in this area could uncover hitherto undiscovered relationships between marketing mix variables using the Lambert W function.

Appendix - A

We derive the optimal prices and market shares for the case involving competition (equations (10) and (11)) and then show how it readily provides the solution in the monopoly scenario (equations (6) and (7)).

Solving for price P_i

From equation (9) in the paper we have the general form of the optimal price shown below,

$$P_i^* = \frac{1 + \beta_i C_i (1 - S_i)}{\beta_i (1 - S_i)} \quad (\text{A-1})$$

Simplifying and rewriting (A-1) we get,

$$P_i^* = \frac{1}{\beta_i (1 - S_i)} + C_i. \quad (\text{A-2})$$

Substituting equation (3) for share from the paper, in equation (A-2) we get,

$$P_i^* = \frac{1}{\beta_i \left(1 - \frac{e^{U_i}}{1 + e^{U_i} + e^{U_k}}\right)} + C_i \quad (\text{A-3})$$

which can be simplified as $P_i^* = \frac{1}{\beta_i} + \frac{e^{\beta_{0i} - \beta_i P_i}}{\beta_i (1 + e^{U_k})} + C_i$. If we let $(1 + e^{U_k}) = \alpha$, then substituting this in the equation for price we find that,

$$P_i^* = \frac{1}{\beta_i} + \frac{e^{\beta_{0i} - \beta_i P_i}}{\beta_i \alpha} + C_i. \quad (\text{A-4})$$

Multiplying (A-4) by β_i and then subtracting β_0 from both sides, we have

$$\beta_i P_i^* - \beta_0 = \frac{e^{\beta_{0i} - \beta_i P_i}}{\alpha} + 1 + \beta_i C_i - \beta_0. \quad (\text{A-5})$$

Rewriting (A-5) we have, $\frac{e^{\beta_{0i} - \beta_i P_i}}{\alpha} - \beta_i P_i^* + \beta_0 = -1 - \beta_i C_i + \beta_0$. Taking exponentials on both sides and then dividing both sides by α gives,

$$e^{\frac{\beta_{0i} - \beta_i P_i}{\alpha}} \cdot \frac{e^{-\beta_i P_i^* + \beta_0}}{\alpha} = \frac{e^{-1 - \beta_i C_i + \beta_0}}{\alpha}. \quad (\text{A-6})$$

Assume $\frac{e^{-\beta_i P_i^* + \beta_0}}{\alpha} = W$, then we rewrite (A-6) as $W e^W = \frac{e^{-1 - \beta_i C_i + \beta_0}}{\alpha}$. This expression is similar to equation for LambertW given by the expression $W e^W = x$, hence the solution is given by

$$W = W\left(\frac{e^{-1 - \beta_i C_i + \beta_0}}{\alpha}\right) \quad (\text{A-7})$$

Substituting for W, we find $\frac{e^{-\beta_i P_i^* + \beta_0}}{\alpha} = W\left(\frac{e^{-1 - \beta_i C_i + \beta_0}}{\alpha}\right)$. Then, using the logarithmic property of the LambertW function (i.e. $\ln(W(x)) = \ln(x) - W(x)$) and taking the natural logarithms on both sides we have

$$-\beta_i P_i^* + \beta_0 - \ln(\alpha) = -1 - \beta_i C_i + \beta_0 - \ln(\alpha) - W\left(\frac{e^{-1 - \beta_i C_i + \beta_0}}{\alpha}\right) \quad (\text{A-8})$$

Equation (A-8) further simplifies to $P_i^* = \frac{1 + \beta_i C_i + W\left(\frac{e^{-1 - \beta_i C_i + \beta_0}}{\alpha}\right)}{\beta_i}$, which is the

closed form solution of P_i^* independent of the effect of its own market share. Substituting for α , we get,

$$P_i^* = \frac{1 + W\left(\frac{e^{(\beta_{0i} - 1 - C_i \beta_i)}}{1 + e^{(\beta_{0k} - P_k \beta_k)}}\right) + C_i \beta_i}{\beta_i}. \quad (\text{A-9})$$

Substituting P_{ic} and C_{ic} for P_i and C_i respectively we get equation (10)

In the case of a monopolist, $\alpha=1$, then equation (A-8) becomes equivalent to equation (6), $P_i^* = \frac{1 + W(e^{\beta_{0i} - 1 - C_i \beta_i})}{\beta_i} + C_i$. Hence, the closed form solutions for

price in the case of competition and monopoly are derived.

Finally we obtain the derivative of price with respect to the model parameters below:

$\frac{\partial P_i^*}{\partial \beta_{0i}} = \frac{1}{\beta_i} \cdot \frac{\partial W(e^{\beta_{0i}-1-C_i\beta_i})}{\partial \beta_{0i}}$ and using the derivative of the Lambert W function as described in Section 2, we obtain $\frac{\partial P_i^*}{\partial \beta_{0i}} = \frac{W(e^{\beta_{0i}-1-C_i\beta_i})}{(1+W(e^{\beta_{0i}-1-C_i\beta_i}))\beta_i}$.

Similarly, we obtain

$$\frac{\partial P_i^*}{\partial \beta_i} = \frac{\beta_i \frac{\partial W(e^{\beta_{0i}-1-C_i\beta_i})}{\partial \beta_i} - 1 + W(e^{\beta_{0i}-1-C_i\beta_i})}{\beta_i^2} = -\frac{C_i W(e^{\beta_{0i}-1-C_i\beta_i})}{(1+W(e^{\beta_{0i}-1-C_i\beta_i}))\beta_i} - \frac{1+W(e^{\beta_{0i}-1-C_i\beta_i})}{\beta_i^2}.$$

Solving for market share S_{it}

As previously noted, we still maintain the notation $(1+e^{U_{kt}}) = \alpha$. Thus substituting this into equation (3) from the paper we have,

$$S_i^* = \frac{e^{\beta_{0i}-\beta_i P_i^*}}{\alpha + e^{\beta_{0i}-\beta_i P_i^*}}. \quad (\text{A-10})$$

Rewriting (A-10) we have, $S_i^* = \frac{e^{\beta_{0i}}}{\alpha e^{\beta_i P_i^*} + e^{\beta_{0i}}}$. Substituting (A-9) in this expression,,

$$S_i^* = \frac{e^{\beta_{0i}}}{\alpha e^{1+\beta_i C_i} e^{W\left(\frac{e^{-1-\beta_i C_i + \beta_{0i}}}{\alpha}\right)} + e^{\beta_{0i}}}. \quad (\text{A-11})$$

Using the LambertW formulation given by $W e^W = x$, we can rewrite (A-11) as

$$S_i^* = \frac{e^{\beta_{0i}}}{\alpha e^{1+\beta_i C_i} \frac{e^{-1-\beta_i C_i + \beta_{0i}}}{\alpha W\left(\frac{e^{-1-\beta_i C_i + \beta_{0i}}}{\alpha}\right)} + e^{\beta_{0i}}}. \quad \text{Upon simplification, it becomes}$$

$$S_i^* = \frac{1}{\frac{1}{W\left(\frac{e^{-1-\beta_i C_i + \beta_{0i}}}{\alpha}\right)} + 1}, \text{ which can be rewritten as } S_i^* = \frac{W\left(\frac{e^{-1-\beta_i C_i + \beta_{0i}}}{\alpha}\right)}{1 + W\left(\frac{e^{-1-\beta_i C_i + \beta_{0i}}}{\alpha}\right)}.$$

Simply substituting for α we get,

$$S_i^* = \frac{W\left(\frac{e^{(\beta_{0i}-1-C_i\beta_i)}}{1+e^{(\beta_{0k}-P_k\beta_k)}}\right)}{1+W\left(\frac{e^{(\beta_{0i}-1-C_i\beta_i)}}{1+e^{(\beta_{0k}-P_k\beta_k)}}\right)}. \quad (\text{A-12})$$

Hence, replacing S_i with S_{ic} and C_i with C_{ic} we derive equation (11). Letting $\alpha=1$ in the case of a monopolist we derive equation (7) which is

$$S_i^* = \frac{W(e^{\beta_{0i}-1-C_i\beta_i})}{1+W(e^{\beta_{0i}-1-C_i\beta_i})}.$$

Appendix – B

Pseudo-code used to calculate the equilibrium prices

1. Substitute the values of the parameters and costs into equations (10) and (12)
2. Evaluate the LambertW function for these values.
3. Solve the two equations (10) and (12) simultaneously to determine the equilibrium prices, give the parameters, costs and the values of the LambertW functions.

Comments:

Step (3) in the pseudo-code was carried out using the ‘fsolve’ function in Matlab, which solves for variables in simultaneous nonlinear equations. The code is available upon request.

References

- Abramson, C., R.L. Andrews, I.S. Currim, M. Jones. 2000. Parameter Bias from Unobserved Effects in the Multinomial Logit Model of Consumer Choice. *Journal of Marketing Research*. 37(November), 410-426.
- Basuroy, S., D. Nguyen. 1998. Multinomial Logit Market Share Models: Equilibrium Characteristics and Strategic Implications. *Management Science*. 44(10), 1396-1408.
- Besanko, D, S. Gupta, D. Jain. 1998. Logit Demand Estimation Under Competitive Pricing Behavior: An Equilibrium Framework. *Marketing Science*. 44(11), 1533-1547.
- Carpenter, G.S., D.R. Lehmann. 1985. A Model of Marketing Mix, Brand Switching, and Competition. *Journal of Marketing Research*. 22(August), 318-329.

- Corless, R.M., D.J. Jeffrey, D.E. Knuth. 1997. A sequence of series for the Lambert W function. *In Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, Maui, Hawaii. New York: ACM Press. 197-204.
- Corless, R.M., G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth. 1996. On the Lambert W function. *Advances in Computational Mathematics*. 5, 329-359.
- Croix, D. de la, O. Licandro. 1999. Life Expectancy and Endogenous Growth. *Economics Letters*. 65, 255-263.
- Euler, L. 1779 (reprinted 1921). De serie Lambertina plurimisque eius insignibus proprietatibus. *In Opera Omnia*. Leipzig and Berlin: B. G. Teubner. 6(1), 350-369.
- Gruca, T.S., D. Sudharshan. 1991. Equilibrium Characteristics of Multinomial Logit Market Share Models. *Journal of Marketing Research*. 18(November), 480-482.
- Gruca, T.S., R.K. Kumar, D. Sudharshan. 1992. An Equilibrium Analysis of Defensive Response to Entry using a Coupled Response Function Model. *Marketing Science*. 11(4), 348-358.
- Guadagni, P.M., J.D.C. Little. 1983. A Logit Model of Brand Choice Calibrated on Scanner Data. *Marketing Science*. 2(3), 203-238. Jeffrey, D.J., D.E.G. Hare, R.M. Corless, 1996. Unwinding the branches of the Lambert W function. *The Mathematical Science*. 21, 1-7.
- Jeffrey, D.J., R.M. Corless, D.E.G Hare, D.E. Knuth. 1995. Sur l'inversion de $y^a e^y$ au moyen des nombres de Stirling associes. *Comptes Rendus de l'Académie des Sciences de Paris t.320, Serie I*. 1449-1452.
- Kamakura, W.A., G.J. Russell. 1989. A Probabilistic Choice Model for Market Segmentation and Elasticity Structuring. *Journal of Marketing Research*. 26(November), 379-90.
- Karnani, A. 1985. Strategic Implications of Market Share Attraction Models. *Management Science*. 31(5), 536-547.
- Kauffman, S., J. Lobo, W.G. Macready. 2000. Optimal Search on a Technology Landscape. *Journal of Economic Behavior and Organization*. 43(2), 141-166.
- Lilien, G.L., A.A. Ruzdic. 1982. Analyzing Neural Experiments in Industrial Markets. *TIMS Studies in Management Sciences*. 241-269.
- Lilien, G.L., P. Kotler. 1983. *Marketing Decision Making: A Model-Building Approach*. Harper Row Publishers, Inc. New York.
- McFadden, D. 1978. Modelling the Choice of Residential Location. A. Karlquist et al., eds. *Spatial Interaction Theory and Residential Location*. North Holland, Amsterdam. 75-96.

- Valluri, S.R., D.J. Jeffrey, R.M. Corless. 2000. Some applications of the Lambert W function to physics. *Canadian Journal of Physics*. 78, 823-831.
- Warburton, R.D.H. 2004. An Analytical Investigation of the Bullwhip Effect. *Production and Operations Management*. 13(2), 150-160.
- Warburton, R.D.H., J. Wang. 2004. Analysis of asymptotic projectile motion with air resistance using Lambert W function. *American Journal of Physics*. November, 72(11), 1404-1407.